Big Goal : study $H^k(GL_n\mathbb{Z};\mathbb{Q})$ for $k,n \geq 0$.

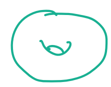
Aim of Talk : Describe recent results establishing Hopf algebra structure on $\bigoplus_{k,n\geq 0} H^k(GL_n\mathbb{Z};\mathbb{Q})$

(I) Group (Co)homology

Q: What is $H_k(\underset{\substack{\uparrow\\ \text{group}}}{G};\underset{\substack{\uparrow\\ G \curvearrowright M \\ M \text{ ab. group}}}{M})$, $H^k(G;M)$?

Topologically: (Co)hom. of "classifying space of G"

$H_k(X_G;M)$, $H^k(X_G;M)$

Eg: $\mathbb{Z} \rightsquigarrow$  S^1 $\quad \mathbb{Z}\times\mathbb{Z} \rightsquigarrow$  $S^1\times S^1$

Algebraically: – "flat resolution of M"

$$\cdots \to F_1 \to F_0 \to M \to 0$$

– Take H_* of

$$\cdots \to F_1 \otimes_G \mathbb{Z} \to F_0 \otimes_G \mathbb{Z} \longrightarrow 0$$

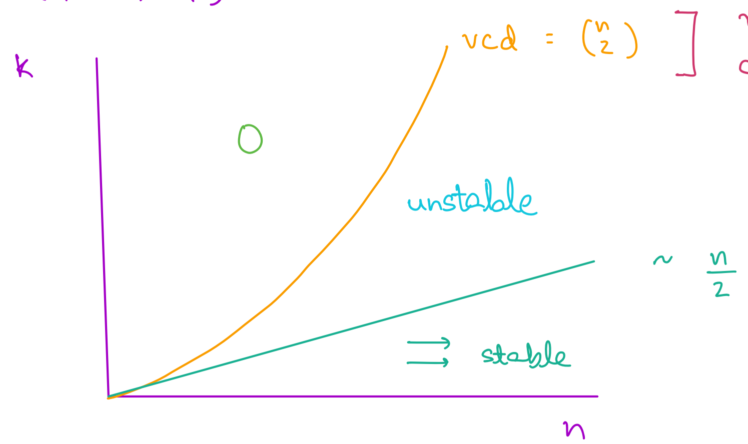
Takeaway: Can define $H_k(G;M)$, $H^k(G;M)$ algebraically and topologically

Our Focus : $H^k(GL_n\mathbb{Z};\mathbb{Q})$ $\quad k,n \geq 0$

(II) The groups $H^k(GL_n\mathbb{Z};\mathbb{Q})$

Important in – Number Theory
K- Theory
Topology

$H^k(GL_n\mathbb{Z};\mathbb{Q})$



] related to fact that $\exists$ classifying space model of dim $\binom{n}{2}$ ... kinda ...

* unstable range mostly unknown ; full table computed only for $n \leq 7$.

(III) Duality for $GL_n\mathbb{Z}$

- $GL_n\mathbb{Z}$ is a rational duality group ; satisfies an analogue of Poincare duality of manifolds

Thm : $H^k(GL_n\mathbb{Z};\mathbb{Q}) \cong H_{\binom{n}{2}-k}(GL_n\mathbb{Z};\ \underbrace{St_n}_{\text{Steinberg module}})$

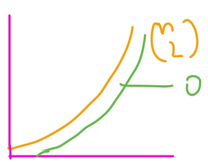
[Borel-Serre, Bieri-Eckmann]

(will define later)

Advantages : • High deg $H^k \rightsquigarrow$ low deg H_*.

Can compute using partial resolutions of St_n

$$F_1 \to F_0 \to St_n \to 0$$



- This approach has been used to show some of these groups are 0 -

• Algebraic Structures on $St_n \rightsquigarrow$ Algebraic structures on $\bigoplus_{k,n\geq 0} H_k(GL_n\mathbb{Z};St_n)$

- Ash-Miller-Patzt (2024) -

Thm: $\bigoplus_{k,n} H_k(GL_n\mathbb{Z};St_n)$ forms a commutative graded Hopf algebra

$$\boxed{H_k(GL_n\mathbb{Z};St_n)\otimes H_\ell(GL_m\mathbb{Z};St_m)\to H_{k+\ell}(GL_{m+n}\mathbb{Z};St_{m+n})}$$

IV The Steinberg Module

- Defined in terms of certain simplicial complexes $\underbrace{T_n\mathbb{Q}}_{\text{Solomon-Tits buildings}}$

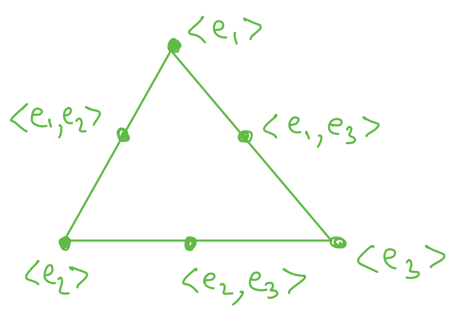
Vertices $\longleftrightarrow$ $0 \subsetneq V \subsetneq \mathbb{Q}^n$ proper, nonzero subspaces

p-simplices $\longleftrightarrow$ Flags of subspaces
$0 \subsetneq V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_p \subsetneq \mathbb{Q}^n$

Eg: n=2 $T_2\mathbb{Q}$ has a vertex for every line $L \subset \mathbb{Q}^2$

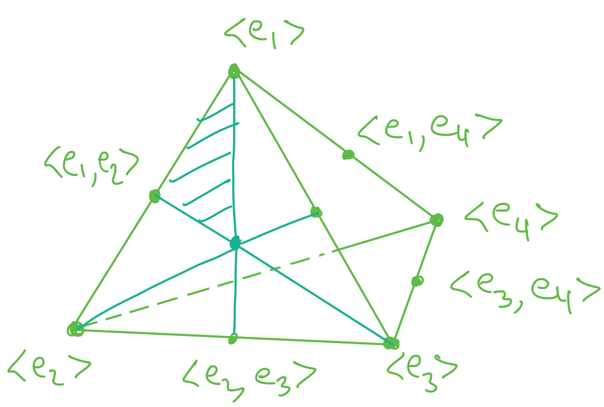
$\langle e_1\rangle \qquad \langle e_2\rangle$] "apartment"

Eg: n=3 $T_3\mathbb{Q}$ has vertices for lines & planes $L, P \subset \mathbb{Q}^3$.
Edges $\leftrightsquigarrow$ inclusions



"apartment"

Eg: n=4



"apartment"

In general, apartment of $\mathcal{T}_n\mathbb{Q} \cong \partial\Delta^{n-1}$ (bdry of $(n-1)$-simplex)

In fact, $\mathcal{T}_n\mathbb{Q}$ is made out of "gluing apartments together" in a certain way.

<u>Thm</u>: $\mathcal{T}_n\mathbb{Q} \simeq \bigvee S^{n-2}$
[Solomon-Tits]

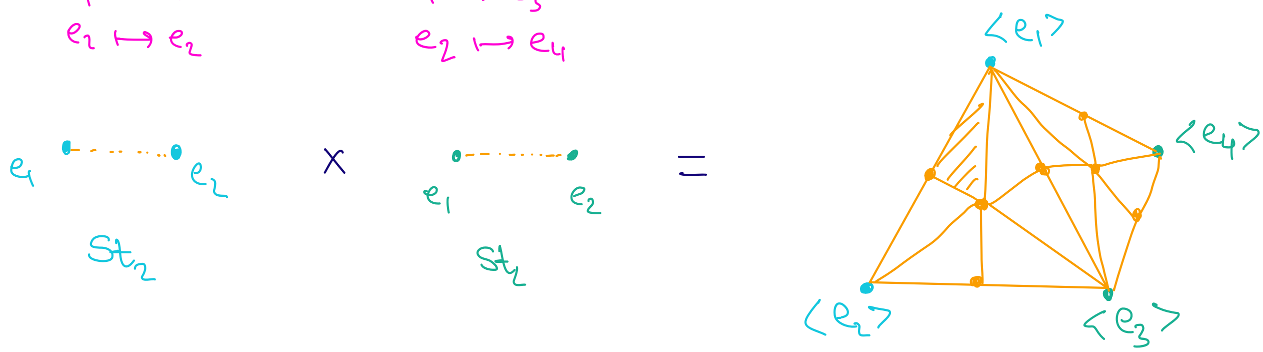
$$\boxed{St_n := \tilde{H}_{n-2}(\mathcal{T}_n\mathbb{Q})} \Big\} \text{ generated by apartment classes}$$

<u>Next Goal</u>: Describe a product and coproduct on $\bigoplus_n St_n$ (analogous to multiplication and factoring in $\mathbb{N}$)

<u>Eg</u>: (Product)

$$\underset{St_2}{[e_1, e_2]} \times \underset{St_2}{[e_1, e_2]} = \underset{St_4}{[e_1, e_2, e_3, e_4]}$$

$$\underset{\substack{e_1 \mapsto e_1 \\ e_2 \mapsto e_2}}{\mathbb{Q}^2} \oplus \underset{\substack{e_1 \mapsto e_3 \\ e_2 \mapsto e_4}}{\mathbb{Q}^2} \cong \mathbb{Q}^4$$



In general, $St_n \otimes St_m \to St_{m+n}$

Formal algebraic results let us get

$$\boxed{H_k(GL_n\mathbb{Z}; St_n) \otimes H_\ell(GL_m\mathbb{Z}; St_m) \to H_{k+\ell}(GL_{m+n}\mathbb{Z}; St_{m+n})}$$

Eg: (Coproduct)

$$[e_1, e_2, e_3, e_4] \longmapsto [e_1, e_2] \otimes [e_3, e_4]$$
$$+ \; [e_2, e_3] \otimes [e_1, e_4]$$
$$+ \; \ldots$$
$$+ \; [e_1] \otimes [e_2, e_3, e_4]$$
$$+ \; \ldots$$

$$St_n \longrightarrow \bigoplus_q St_q \otimes St_{n-q}$$

We get

$$\boxed{H_k(GL_n\mathbb{Z}; St_n) \longrightarrow \bigoplus_{p,q} H_p(GL_q\mathbb{Z}; St_q) \otimes H_{k-p}(GL_{n-q}\mathbb{Z}; St_{n-q})}$$

(V) Hopf Algebras

The product & coproduct on $\bigoplus H_k(GL_n\mathbb{Z}; St_n)$ are compatible in the sense of a "Hopf Algebra"

Analogous to the compatibility of multiplying and factoring in $\mathbb{N}$.

Defn: H is a Hopf algebra if $\exists$ $\nabla : H\otimes H \to H$
$\Delta : H \to H \otimes H$

s.t.

$$\begin{tikzcd}
H\otimes H \arrow[r] \arrow[d] & H \arrow[r] \arrow[d, phantom, "\circlearrowright"] & H\otimes H \\
(H\otimes H)\otimes(H\otimes H) \arrow[rr] & & (H\otimes H)\otimes(H\otimes H) \arrow[u]
\end{tikzcd}$$

Thm [Milnor-Moore]: A graded commutative Hopf algebra is freely generated by its indecomposables.

Ash-Miller-Patzt:

- Showed $\bigoplus_{k,n} H_k(GL_n\mathbb{Z}; St_n)$ is a graded commutative Hopf algebra
- Found some indecomposables, and used them to get new H_*-classes by multiplying them.

VI $H^*(Sp_{2n}\mathbb{Z}; \mathbb{Q})$

- also of interest
- analogous duality theorems St^{ω}_n
- trying to get Hopf module structure on $H_*(Sp_{2n}\mathbb{Z}; St^{\omega}_n)$